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THE C-COMPLEX CLASP NUMBER OF LINKS

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In the 1980s, Daryl Cooper introduced the notion of a C-complex (or clasp-complex) bounded by a link and explained how to compute signatures and polynomial invariants using a C-complex. Since then, this has been extended by works of Cimasoni, Florens, Mellor, Melvin, Conway, Toffoli, Friedl, and others to compute other link invariants. Informally, a C-complex is a union of surfaces which are allowed to intersect each other in clasps. We study the minimal number of clasps amongst all C-complexes bounded by a fixed link L. This measure of complexity is related to the number of crossing changes needed to reduce L to a boundary link. We prove that if L is a 2-component link with nonzero linking number, then the linking number determines the minimal number of clasps amongst all C-complexes. In the case of 3-component links, the triple linking number provides an additional lower bound on the number of clasps in a C-complex.

1. Introduction

There is a generalization of a Seifert surface to the setting of links called a *C-complex* or *clasp-complex* originally defined by Cooper [4; 3]. Informally, if $L = L_1 \cup \cdots \cup L_n$ is an n-component link, then a C-complex for L is a collection of Seifert surfaces $F = F_1 \cup \cdots \cup F_n$ for the components of L which are allowed to intersect, but only in clasps. A local picture of a clasp appears in Figure 1. Figures 3 and 4 depict examples of C-complexes. The precise definition of a C-complex appears later in Definition 8.

If a C-complex F for L has no clasp intersections, then F is a collection of disjoint Seifert surfaces for the components of L. In this case L is called a *boundary link* and F is called a *boundary surface*. Thus, the number of clasps in a C-complex can be used to measure how far F is from being a boundary surface and so how far L is from being a boundary link. In this paper we shall study the minimal number of clasps amongst all C-complexes bounded by L. This should not be confused with the clasp number introduced by Shibuya in [10].

Definition 1. For a link L we define the *clasp number* of L, denoted by $C(L)$, to be the minimum number of clasps amongst all C-complexes bounded by L.

For a 2-component link $L = L_1 \cup L_2$ and any C-complex $F = F_1 \cup F_2$ bounded by L, the linking number, denoted by $\mathrm{lk}(L_1, L_2)$, can be computed as the number of positive clasps in F minus the number of negative. It follows that $C(L) \geq |\mathrm{lk}(L_1, L_2)|$. Our first main result is that for most 2-component links, $C(L) = |\mathrm{lk}(L_1, L_2)|$.

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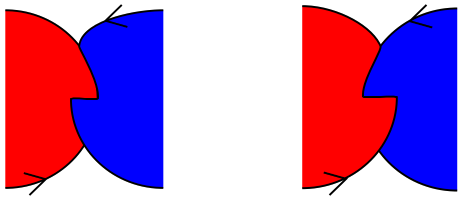

Figure 1. A positive clasp, left, and a negative clasp, right.

Theorem 2. *Let* $L = L_1 \cup L_2$ *be a* 2*-component link. If* $\operatorname{lk}(L_1, L_2) \neq 0$, *then* $C(L) = |\operatorname{lk}(L_1, L_2)|$. *If* $\operatorname{lk}(L_1, L_2) = 0$, *then* $C(L) \in \{0, 2\}$.

We mentioned that the number of clasps in a C-complex for L measures how far L is from being a boundary link. We take a moment and make that explicit. Any link can be reduced to a boundary link by a finite sequence of crossing changes. Indeed, that boundary link can be taken to be the unlink. Let $B(L)$ be the minimum number of crossing changes needed to reduce L to a boundary link. If F is a C-complex for L admitting $C(L)$ total clasps, then by changing a crossing at each clasp, as in Figure 2, one reduces F to a boundary surface and so L to a boundary link. Therefore

$$B(L) \leq C(L).$$

On the other hand, changing a crossing of L changes at most one linking number of L and that by at most 1. As any boundary link has vanishing pairwise linking numbers, we conclude that if $L = L_1 \cup \cdots \cup L_n$ is an n-component link, then

$$\sum_{1 \leq i < j \leq n} |\operatorname{lk}(L_i, L_j)| \leq B(L).$$

By Theorem 2, if $L = L_1 \cup L_2$ has only two components and $\operatorname{lk}(L_1, L_2) \neq 0$, then $C(L) = |\operatorname{lk}(L_1, L_2)|$. Thus, in this case we have $|\operatorname{lk}(L_1, L_2)| \leq B(L) \leq C(L) = |\operatorname{lk}(L_1, L_2)|$. We arrive at the following corollary.

Corollary 3. *Let* $L = L_1 \cup L_2$ *be a* 2*-component link. If* $\operatorname{lk}(L_1, L_2) \neq 0$, *then there exists a sequence of* $|\operatorname{lk}(L_1, L_2)|$ *crossing changes reducing L to a boundary link. If* $\operatorname{lk}(L_1, L_2) = 0$, *then either L is a boundary link or there exists a sequence of at most* 2 *crossing changes reducing L to a boundary link.*

Example 4. In order to illustrate Theorem 2 and Corollary 3, consider the link of Figure 3. The depicted C-complex has three clasps. Since $\operatorname{lk}(L_1, L_2) = 1$, there exists a C-complex bounded by L with a single clasp and, perhaps more surprisingly, there exists a single crossing change reducing L to a boundary link.

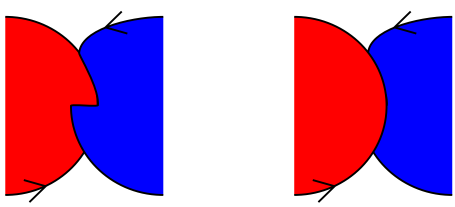

Figure 2. A clasp, left, and a crossing change removing the clasp, right.

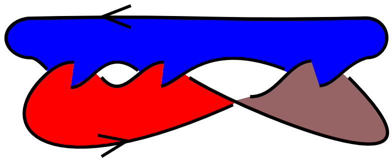

Figure 3. A 2-component link with linking number 1.

According to Theorem 2, the linking number completely determines the clasp number of 2-component links with nonzero linking number, and the clasp number of 2-component links with linking number zero is bounded. This behavior does not extend to links of more than 2 components. In [8], Milnor introduced a family of higher-order linking invariants. The first of these is called the triple linking number and is denoted by μ_{ijk}. It is well defined when the pairwise linking numbers vanish and measures the linking of the i-th, j-th, and k-th components. According to Mellor and Melvin [7], $\mu_{ijk}(L)$ can be computed in terms of the clasps of a C-complex bounded by L. Thus, it comes as no surprise that $\mu_{123}(L)$ can be used to deduce a bound on $C(L)$. We explicitly compute this bound.

Theorem 5. *Let $L = L_1 \cup L_2 \cup L_3$ be a 3-component link with vanishing pairwise linking numbers. Then $C(L) \geq 2\lceil 2\sqrt{|\mu_{123}(L)|/3}\rceil$. Here $\lceil - \rceil$ is the ceiling function.*

In order to illustrate the power of this theorem, we compute the clasp number of some examples. The Borromean rings, denoted by BR, has $\mu_{123}(\mathrm{BR}) = 1$ and so by Theorem 5, $C(\mathrm{BR}) \geq 4$. The left-hand side of Figure 4 depicts a C-complex bounded by BR with four clasps. Thus, $C(\mathrm{BR}) = 4$. For any $n \in \mathbb{N}$, the generalized Borromean rings BR^n of the right-hand side of Figure 4 bound a C-complex with $4n$ clasps, and $\mu_{123}(\mathrm{BR}^n) = n^2$. We do this computation in Example 10. As a consequence, we get the following corollary, producing links with vanishing pairwise linking numbers and arbitrarily large clasp number.

Corollary 6. *For any $n \in \mathbb{N}$, consider the generalized Borromean rings BR^n of Figure 4, right. The pairwise linking numbers of BR^n vanish and yet $2\lceil 2n/\sqrt{3}\rceil \leq C(L) \leq 4n$.*

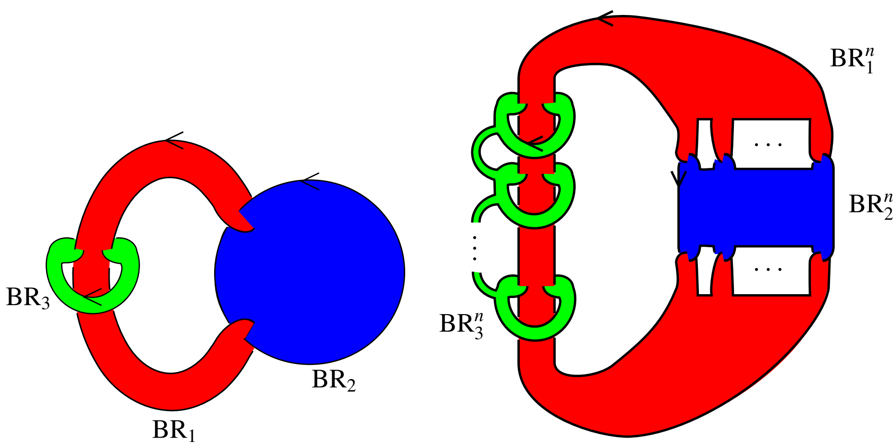

Figure 4. A C-complex bounded by the Borromean rings, left, and a C-complex bounded by the generalized Borromean rings BR^n, right.

In [7], Mellor and Melvin provided a means of computing $\mu_{123}(L)$ in terms of any collection of Seifert surfaces for the components of L. We shall use this result in the special case of a C-complex. While a more complete description appears in Section 3, we recall it informally now. Start with a link $L = L_1 \cup L_2 \cup L_3$ bounding a C-complex $F = F_1 \cup F_2 \cup F_3$, follow a component L_k of L, and record a word $w_k(F)$ in $x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}$ capturing the order and sign of the clasps L_k encounters. Set $e_{ij}(w_k(F)) \in \mathbb{Z}$ to be the signed count of the number of the x_i appearing in w_k before an x_j. The triple linking number is given by $\mu_{123}(L) = e_{12}(w_3(F)) + e_{23}(w_1(F)) + e_{31}(w_2(F))$.

A technical result we use in our proof of Theorem 5 is a new geometric strategy to compute $e_{ij}(w)$. For any word w in letters $x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}$ and any $i, j \in \{1, 2, 3\}$, construct a piecewise linear curve $\gamma_{ij}(w)$ in $\mathbb{R}^2$ as follows. Start at the origin $(0, 0)$. Each time you see an x_i (respectively x_i^{-1}, x_j, x_j^{-1}) in w, travel right (respectively left, up, down) a length of 1. The following reveals that $e_{ij}(w)$ is the area enclosed by this curve.

Proposition 7. *Let $w = \prod_{n=1}^{m} x_{i_n}^{\epsilon_n}$ be a word in letters $x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}$. For any $i \neq j \in \{1, 2, 3\}$,*

$$e_{ij}(w) = \oint_{\gamma_{ij}(w)} x\, dy.$$

Additionally, if $\gamma_{ij}(w)$ is a simple closed curve with counterclockwise orientation, then $e_{ij}(w)$ is the area enclosed by $\gamma_{ij}(w)$.

1.1. Questions. Theorem 2 states that any 2-component link with nonzero linking number has a C-complex admitting precisely $|\mathrm{lk}(L_1, L_2)|$ clasps. However, our proof makes no attempt to minimize the first Betti number of the C-complex, which is the measure of complexity most directly accessible using the tools like Alexander polynomial or signature [1; 2]. We pose the following question.

Question 1. Suppose that $L = L_1 \cup L_2$ is a 2-component link with nonzero linking number. Amongst all C-complexes F bounded by L admitting precisely $|\mathrm{lk}(L_1, L_2)|$ clasps, what is the minimal value for $\beta_1(F)$? Is it possible to simultaneously minimize the number of clasps in F as well as $\beta_1(F)$?

Theorem 2 almost completely determines $C(L)$ for 2-component links. Theorem 5 concludes that $C(L) \geq 2\lceil 2\sqrt{|\mu_{123}(L)|/3}\rceil$ for three component links with vanishing linking numbers. One might ask if equality holds.

Question 2. Let $L = L_1 \cup L_2 \cup L_3$ be a 3-component link with vanishing pairwise linking numbers and $\mu_{123}(L) \neq 0$. Does it follow that $C(L) = 2\lceil 2\sqrt{|\mu_{123}(L)|/3}\rceil$?

More specifically, for any $n \in \mathbb{N}$, consider the generalized Borromean rings BR^n of the right-hand side of Figure 4. Corollary 6 concludes that $2\lceil 2n/\sqrt{3}\rceil \leq C(\mathrm{BR}^n) \leq 4n$. When $n = 2$ this gives $6 \leq C(\mathrm{BR}^2) \leq 8$.

Question 3. What is $C(\mathrm{BR}^n)$?

Additionally, one might ask about the clasp number of links of more than three components.

Question 4. Let $n \geq 3$ and let $L = L_1 \cup \cdots \cup L_n$ be an n-component link with vanishing pairwise linking numbers and $\mu_{ijk}(L) \neq 0$ for some i, j, k. Is there a formula for $C(L)$ in terms of the set of all triple linking numbers of L?

In the case of links of more than 2 components with nonvanishing pairwise linking numbers, the triple linking numbers are not well defined. Instead, by [5, Theorem 1.1] there is a *total triple linking number* recording all of the individual triple linking numbers taking values in the quotient of $\mathbb{Z}^{\binom{n}{3}}$ by a subgroup depending on the individual linking numbers [5, Definition 5.7].

Question 5. Let $L = L_1 \cup \cdots \cup L_n$ be an n-component link with either a nonvanishing pairwise linking number or nonvanishing total triple linking number. Is there a formula for $C(L)$ in terms of the linking numbers and the total triple linking number?

2. C-complexes and the proof of Theorem 2

Throughout this paper all knots are smoothly embedded curves in S^3, and all surfaces are smoothly embedded in S^3, compact, connected, and oriented. A smoothly embedded compact oriented surface with boundary equal to a knot K is called a *Seifert surface* for K. We begin by recalling the formal definition of a C-complex.

Definition 8. [2, Section 2.1] Given a link $L = L_1 \cup \cdots \cup L_n$, a C-complex for L is a collection of Seifert surfaces $F = F_1 \cup \cdots \cup F_n$ for the components of L, which may intersect transversely with the following constraints:

(1) For each $i, j \in \{1, \dots, n\}$, $F_i \cap F_j$ is a union of simple arcs running from a point in $L_i = \partial F_i$ to a point in $L_j = \partial F_j$. These arcs are called *clasps*; see Figure 1.

(2) $F_i \cap F_j \cap F_k = \varnothing$ for any three distinct i, j, k.

The fact that F_i and F_j intersect transversely implies that at every $p \in L_i \cap F_j$, the tangent vector to L_i does not lie in the tangent plane of F_j. Since F_i is oriented, there is a preferred choice of normal vector to F_i at every point in F_i. We call a point of intersection $p \in L_i \cap F_j$ *positive* if the dot product of the tangent vector to L_i at p with the normal vector to F_j at p is positive. If the dot product is negative, then the point of intersection is called *negative*.

A clasp between F_i and F_j has endpoints given by a point in $L_i \cap F_j$ and a point in $L_j \cap F_i$. We call a clasp *positive* (or *negative*, respectively) if these points of intersection are positive (or negative, respectively). Local pictures of a positive and a negative clasp appear in Figure 1.

For the remainder of this section we restrict to the case that the number of components is $n = 2$. If F_2 is any Seifert surface for L_2, then the linking number $\operatorname{lk}(L_1, L_2)$ is given by counting with sign how many times L_1 intersects F_2; see [9, Section 5D]. If $F_1 \cup F_2$ is a C-complex for $L_1 \cup L_2$, then this is precisely the same as the signed count of the clasps shared by F_1 and F_2. We are now ready to prove Theorem 2.

Theorem 2. *Let* $L = L_1 \cup L_2$ *be a* 2*-component link. If* $\operatorname{lk}(L_1, L_2) \neq 0$, *then* $C(L) = |\operatorname{lk}(L_1, L_2)|$. *If* $\operatorname{lk}(L_1, L_2) = 0$, *then* $C(L) \in \{0, 2\}$.

Proof of Theorem 2. Let $L = L_1 \cup L_2$ be a 2-component link and $F = F_1 \cup F_2$ be any C-complex bounded by L. Let c_+ be the number of positive clasps in F and c_- be the number of negative clasps. By the triangle inequality,

$$|\operatorname{lk}(L_1, L_2)| = |c_+ - c_-| \leq c_+ + c_-,$$

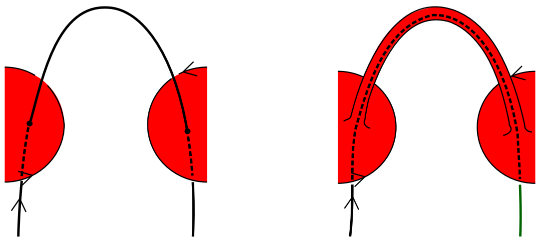

Figure 5. Left, a knot L_2 intersecting an oriented surface F_1 in a positive point of intersection followed by a negative point of intersection. Right, adding a tube to F_1 removes both intersection points.

so that F has at least $|\text{lk}(L_1, L_2)|$ many clasps. As F is an arbitrary C-complex bounded by L, we get that $C(L) \geq |\text{lk}(L_1, L_2)|$. Thus, we need only to show that $C(L) \leq |\text{lk}(L_1, L_2)|$. Since $C(L)$ is the minimum number of clasps amongst all C-complexes bounded by L, it suffices to exhibit a C-complex with precisely $|\text{lk}(L_1, L_2)|$ clasps or 2 clasps in the case that $\text{lk}(L_1, L_2) = 0$. Without loss of generality we shall assume that $\text{lk}(L_1, L_2) \geq 0$.

We begin by producing a pair of Seifert surfaces F_1 and F_2 for L_1 and L_2 which have no negative clasps in their intersection but which may have some non-clasp intersections. Let F_1 be any Seifert surface for L_1. Suppose F_1 is transverse to L_2 and $F_1 \cap L_2$ contains n_+ positive points of intersection and n_- points of negative intersection. If both n_+ and n_- are nonzero, then as you follow L_2 you will at some point encounter a positive point of intersection with F_1 followed by a negative, as in the left side of Figure 5. By adding a tube to F_1 as in the right side of Figure 5, we see a new Seifert surface bounded by L_1, which intersects L_2 in two fewer points. Iterating, we see a Seifert surface for L_1, which we persist in calling F_1, bounded by L_1, which either intersects L_2 in only positive points or only negative points of intersection. Thus, $n_+ = 0$ or $n_- = 0$. Since $n_+ - n_- = \text{lk}(L_1, L_2) \geq 0$ by assumption, we see that $n_- = 0$. By the same process, we find a Seifert surface F_2 which intersects L_1 in only positive points of intersection.

There is no reason to expect that $F_1 \cup F_2$ is a C-complex. After a small isotopy of F_1 and F_2 we may assume that they intersect transversely. Therefore $F_1 \cap F_2$ consists of a collection of

- arcs with one endpoint in $L_1 = \partial F_1$ and the other in $L_2 = \partial F_2$,
- arcs with both endpoints in $L_1 = \partial F_1$ or both endpoints in $L_2 = \partial F_2$, and
- simple closed curves interior to F_1 and interior to F_2.

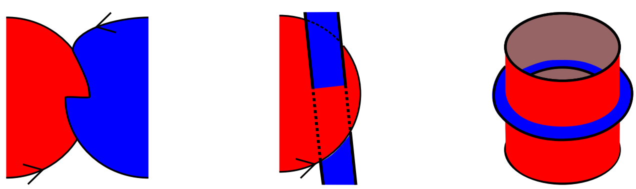

Figure 6. A positive clasp intersection, left, a ribbon intersection, center, and a loop intersection, right.

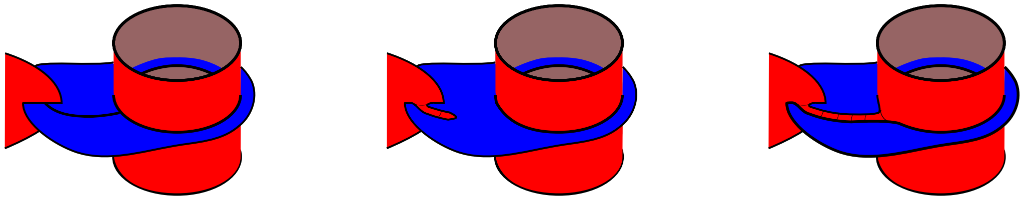

Figure 7. Left, a pair of surfaces sharing a clasp and a loop intersection together with an arc running from the clasp to the loop. Center, performing a finger move to push the clasp intersection closer to the loop. Right, tubing the clasp into the loop results in a single clasp intersection.

We call these types of intersections clasps, ribbons, and loops, respectively. Examples of each appear in Figure 6. Since F_1 has no negative points of intersection with L_2, there can be no negative clasps in $F_1 \cap F_2$. The endpoints of a ribbon intersection are intersection points between F_1 and L_2 (or F_2 and L_1) with opposite signs. Since we have already arranged that there are no negative points of intersection, there can be no ribbon intersections in $F_1 \cap F_2$. Thus, $F_1 \cap F_2$ consists only of loops and positive clasps. It remains to further modify F_1 and F_2 to eliminate all loops.

Assume that $\mathrm{lk}(L_1, L_2) \neq 0$, so that there is at least one clasp in $F_1 \cap F_2$. Let c be one such clasp. Suppose that there exists a loop intersection $\ell \subseteq F_1 \cap F_2$. By taking an outermost loop in F_2 we may arrange that there exists an arc α in F_2 running from a point in c to a point in ℓ. Moreover, we may assume that α connects two points pushed off from F_1 in the same normal direction. Figure 7 reveals how one may add a tube to F_1 following α to combine c and ℓ into a single simple arc. This arc has one endpoint in L_1 and the other in L_2. In other words, it is a clasp. Thus, we have reduced the number of loop intersections by 1 and preserved the number of clasp intersections. Iterating, we eliminate all loop intersections and produce a C-complex for $L = L_1 \cup L_2$ with number of clasps equal to $\mathrm{lk}(L_1, L_2)$, as claimed.

In the case that the linking number is zero, $F_1 \cap F_2$ contains no clasps. If $F_1 \cap F_2$ also has no loops, then $F_1 \cup F_2$ is a C-complex with no clasps and $C(L) = 0$. Otherwise, modify $F_1 \cup F_2$ as in Figure 8 to add a positive and a negative clasp. Now we use the move of Figure 7 just as in the previous paragraph to remove all loop intersections and produce a C-complex with precisely 2 clasps, so $0 \leq C(L) \leq 2$. In order to see that $C(L)$ cannot be 1, notice that since $c_+ - c_- = \mathrm{lk}(L_1, L_2) = 0$, it must be that $c_+ = c_-$. In particular, F has an even number of clasps. This completes the proof. □

Before we begin our analysis of the triple linking number and links of three or more components, we take a moment and explain why the proof of Theorem 2 fails for links of three or more components. One may run the argument of Theorem 2 on a 3-component link $L_1 \cup L_2 \cup L_3$ in order to produce Seifert

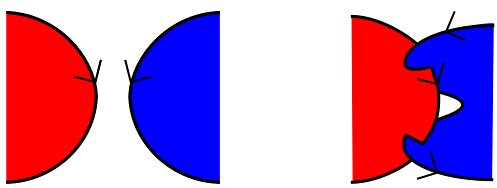

Figure 8. Modifying a C-complex by inserting two canceling clasps.

surfaces F_1, F_2, and F_3 so that for all $i, j \in \{1, 2, 3\}$, $F_i \cap F_j$ consists of only positive or only negative clasps, or when $\mathrm{lk}(L_i, L_j) = 0$ consists of a single positive and a single negative clasp. However, there is no reason for $F_1 \cap F_2 \cap F_3$ to be empty, as is required of a C-complex. Indeed, if L has vanishing linking numbers and $\mu_{123}(L)$ is large, then Theorem 5 implies that there is no way to remove these triple intersections without introducing a large number of clasps.

3. Triple linking numbers via clasps and polyominos

In this section we recall an invariant of links called the triple linking number and provide a formula in terms of the area of a polyomino. A *polyomino* is a region of $\mathbb{R}^2$ consisting of a union of closed unit squares with vertices at points in $\mathbb{Z}^2$.

In [7] Mellor and Melvin produce a formula for the triple linking number from any union of Seifert surfaces for the components of L. We shall recall it in the special case of a C-complex. Let $L = L_1 \cup \cdots \cup L_n$ be an n-component link and $F = F_1 \cup \cdots \cup F_n$ be a C-complex bounded by L. We associate to each $k = 1, \ldots, n$ a word $w_k(F)$ called a *clasp word*, as follows. Pick a basepoint p_k on L_k and follow L_k in the positive direction starting at p_k. Record an x_j whenever L_k crosses through F_j at a positive clasp and x_j^{-1} when L_k crosses F_j at a negative clasp. Let $e_{ij}(w_k(F))$ be given by counting with sign how often in $w_k(F)$ the symbol x_i appears before x_j. More formally, if $w_k(F) = \prod_{v=1}^{m} x_{i_v}^{\epsilon_v}$, then

$$e_{ij}(w_k(F)) = \sum_{v=1}^{m}\sum_{u=1}^{v} \delta(i_u, i)\delta(i_v, j)\epsilon_u\epsilon_v, \tag{1}$$

where we have used the Kronecker δ,

$$\delta(a, b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

We encourage the reader to take a moment and use this definition to compute $e_{12}(x_1x_2x_1^{-1}x_2^{-1}) = 1$. The *triple linking number* is given by

$$\mu_{ijk}(L) = e_{ij}(w_k(F)) + e_{jk}(w_i(F)) + e_{ki}(w_j(F)).$$

When L is a link with vanishing pairwise linking numbers, $\mu_{ijk}(L)$ is independent of the choice of F and of the choice of basepoints.

Example 9. For the sake of clarity, we provide an example computing the triple linking number of the Borromean rings $\mathrm{BR} = \mathrm{BR}_1 \cup \mathrm{BR}_2 \cup \mathrm{BR}_3$ using the C-complex F of Figure 4.

- Following BR_1 starting at the arrow we encounter, in order, a negative clasp with F_3, a positive clasp with F_2, a positive clasp with F_3 and a negative clasp with F_2. Therefore,

$$w(F_1) = x_3^{-1}x_2x_3x_2^{-1}.$$

  Similarly, $w(F_2) = x_1^{-1}x_1$ and $w(F_3) = x_1^{-1}x_1$.
- Count with sign how many times you see x_2 before x_3 in $w(L_1)$ to get $e_{23}(w_1(F)) = +1$. Similarly, $e_{12}(w(F_3)) = e_{31}(w(F_2)) = 0$.

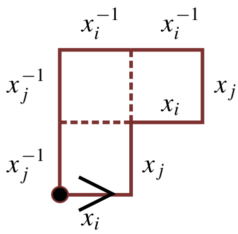

Figure 9. The curve $\gamma_{ij}(w)$ associated to the word $w = x_i x_j x_i x_j x_i^{-2} x_j^{-2}$, together with the region $\gamma_{ij}(w)$ encloses.

- The triple linking number is given by summing,

$$\mu_{123}(\mathrm{BR}) = e_{12}(w_3(F)) + e_{23}(w_1(F)) + e_{31}(w_2(F)) = 1.$$

Our next goal is the statement and proof of Proposition 7, which computes $e_{ij}(w_k(F))$ in terms of some curve $\gamma_{ij}(w_k(F))$ in the plane. We begin by explaining the construction of $\gamma_{ij}(w_k(F))$. Let w be any word in the letters $x_1^{\pm 1}, \dots, x_n^{\pm 1}$. We give a procedure which associates to w a curve in the plane. Start at the point $(0, 0) \in \mathbb{R}^2$. Each time you encounter x_i in w travel right a length of 1. When x_i^{-1} is encountered travel left. When x_j or x_j^{-1} is encountered travel up or down, respectively. Call the resulting curve $\gamma_{ij}(w)$. For instance, when $w = x_i x_j x_i x_j x_i^{-2} x_j^{-2}$, the curve $\gamma_{ij}(w)$ appears in Figure 9. The assiduous reader will now compute $e_{ij}(w) = 3$ using equation (1), which suggestively agrees with the area of the region enclosed by $\gamma_{ij}(w)$.

We are now ready to prove Proposition 7.

Proposition 7. *Let $w = \prod_{v=1}^{m} x_{i_v}^{\epsilon_v}$ be a word in letters $x_1^{\pm 1}, \dots, x_n^{\pm 1}$. For any $i \neq j \in \{1, \dots, n\}$,*

$$e_{ij}(w) = \oint_{\gamma_{ij}(w)} x\, dy.$$

Additionally, if $\gamma_{ij}(w)$ is a simple closed curve with counterclockwise orientation, then $e_{ij}(w)$ is the area enclosed by $\gamma_{ij}(w)$.

Proof of Proposition 7. Let $w = \prod_{v=1}^{m} x_{i_v}^{\epsilon_v}$ be a word in the letters $x_1^{\pm}, \dots, x_n^{\pm 1}$. Let $|w| = m$ be the length of w. Then $\gamma_{ij}(w)$ consists of a concatenation of $|w|$ many curves, $\gamma_{ij}^1(w), \dots, \gamma_{ij}^m(w)$, where $\gamma_{ij}^v(w)$ is constant if $i_v \notin \{i, j\}$ and is a length 1 line segment traveling in a cardinal direction otherwise. Therefore, the integral in question breaks up as

$$\oint_{\gamma_{ij}(w)} x\, dy = \sum_{v=1}^{m} \left(\oint_{\gamma_{ij}^v(w)} x\, dy \right).$$

If $i_v \neq j$ then $\gamma_{ij}^v(w)$ is either constant or parametrizes a horizontal line segment. In either case $dy = 0$ so that $\oint_{\gamma_{ij}^v(w)} x\, dy = 0$. If $i_v = j$ then $\gamma_{ij}^v(w)$ is a vertical line segment parametrized by $\gamma_{ij}^v(t) = (x, t \cdot \epsilon_v + c)$ with x and c constants and t running from 0 to 1. In particular, $dy = \epsilon_v\, dt$. The fixed x-coordinate over which this vertical line sits is the signed count of $u < v$ with $i_u = i$,

$$x = \sum_{u=1}^{v} \delta(i_u, i)\epsilon_u.$$

Thus, in the case that $i_v = j$, we have

$$\oint_{\gamma_{ij}^v(w)} x\,dy = \int_0^1 x \cdot \epsilon_v\,dt = x \cdot \epsilon_v = \sum_{u=1}^{v} \delta(i_u, i)\epsilon_u\epsilon_v.$$

Combining the cases $i_v = j$ and $i_v \neq j$, we see that for all v,

$$\oint_{\gamma_{ij}^v(w)} x\,dy = \delta(i_v, j)\sum_{u=1}^{v} \delta(i_u, i)\epsilon_u\epsilon_v.$$

Summing over all values of v,

$$\oint_{\gamma_{ij}(w)} x\,dy = \sum_{v=1}^{m} \delta(i_v, j)\sum_{u=1}^{v} \delta(i_u, i)\epsilon_u\epsilon_v.$$

An application of the distributive law reduces this to the definition of $e_{ij}(w)$ appearing in equation (1). This completes the proof of the first claim.

The second claim follows from a standard application of Green's theorem. □

Example 10. We now spare a moment for a computation. For any $n \in \mathbb{N}$, consider the generalized Borromean rings BR^n of Figure 4. Using the C-complex of the right-hand side of Figure 4 we get clasp words

$$w_1(F) = x_3^{-n}x_2^n x_3^n x_2^{-n}, \quad w_2(F) = x_1^n x_1^{-n}, \quad w_2(F) = (x_1 x_1^{-1})^n.$$

While one might now use equation (1) to directly compute $e_{12}(w_3)$, $e_{23}(w_1)$, and $e_{31}(w_2)$, we shall use Proposition 7. The curve $\gamma_{23}(w_1(F))$ traces a counterclockwise $n \times n$ square so that $e_{23}(\omega_1) = n^2$. The curve $\gamma_{31}(w_2(F))$ lies in the vertical line $x = 0$ so that $e_{31}(w_2(F)) = 0$. Finally, $\gamma_{23}(w_1(F))$ lies in the horizontal line $y = 0$ so that $e_{12}(w_3(F)) = 0$. Therefore, $\mu_{123}(\mathrm{BR}^n) = n^2$.

4. The proof of Theorem 5

We now turn our attention to a lower bound on the number of clasps in a C-complex in terms of the triple linking number. Notice that the curve $\gamma_{ij}(w(L_k))$ of Section 3 has length equal to the number of clasps in $F_k \cap F_i$ plus the number of clasps in $F_k \cap F_j$. By Proposition 7, $\oint_{\gamma_{ij}(w)} x\,dy = e_{ij}(w(L_k))$. Thus, we begin the proof of Theorem 5 by studying how $\oint_\gamma x\,dy$ provides a lower bound on the length of γ.

For the lemma below, a *polyomino curve* is a closed curve in $\mathbb{R}^2$ given by a concatenation of straight lines of length 1 between points in $\mathbb{Z}^2$. The length of a curve γ is denoted by $\|\gamma\|$.

Lemma 11. *Let γ be a polyomino curve in $\mathbb{R}^2$. Let $A = \oint_\gamma x\,dy$. Then $\|\gamma\| \geq 2\lceil 2\sqrt{|A|}\rceil$.*

Proof. Let γ be a polyomino curve in $\mathbb{R}^2$ and let $A = \oint_\gamma x\,dy$. If γ is a simple closed curve, then a standard application of Green's theorem shows that

$$|A| = \iint_A 1\,dx\,dy$$

is the area of the region R enclosed by γ. In [6], Harary and Harborth showed that the minimum perimeter amongst all polyominos with a fixed area $|A|$ is given by $2\lceil 2\sqrt{|A|}\rceil$. As $\|\gamma\|$ is the perimeter of A, $\|\gamma\| \geq 2\lceil 2\sqrt{|A|}\rceil$, as the lemma claims.

It remains to deal with the case that γ is not simple. Recall that by assumption, γ consists of a concatenation of vertical and horizontal line segments of length 1. Denote the rightward pointing horizontal line segments as $\gamma_1^r(t), \ldots, \gamma_h^r(t)$, the leftward pointing as $\gamma_1^\ell(t), \ldots, \gamma_h^\ell(t)$, the upward as $\gamma_1^u(t), \ldots, \gamma_v^u(t)$ and the downward as $\gamma_1^d(t), \ldots, \gamma_v^d(t)$. As γ is a closed curve, the number of rightward and leftward pointing segments must be equal to each other, as must be the number of upward and downward pointing segments.

Up to a translation and a reparametrization preserving $\|\gamma\|$ and $\oint_\gamma x\,dy$, we may assume that γ is parametrized by some $(x(t), y(t))$ such that the minimum value of $x(t)$ is $x(0) = 0$. It follows for all t that $0 \le x(t) \le h$, where h is the number of rightward-pointing length 1 line segments in γ. Breaking the integral up as a sum,

$$A = \oint_\gamma x\,dy = \sum_{i=1}^{v} \oint_{\gamma_i^u} x\,dy + \sum_{i=1}^{v} \oint_{\gamma_i^d} x\,dy + \sum_{i=1}^{h} \oint_{\gamma_i^r} x\,dy + \sum_{i=1}^{h} \oint_{\gamma_i^\ell} x\,dy. \tag{2}$$

Since γ_i^ℓ and γ_i^r are horizontal line segments, they each have $dy = 0$, so that $\oint_{\gamma_i^r} x\,dy = \oint_{\gamma_i^\ell} x\,dy = 0$. Since γ_i^u is an upward-pointing length 1 line segment, we may parametrize γ_i^u as $(x, t+c)$ where x and c are constant and t runs from 0 to 1. Therefore, $dy = dt$ and $0 \le x \le h$. Thus, $\oint_{\gamma_i^u} x\,dy = \int_0^1 x\,dt = x$ and in particular $0 \le \oint_{\gamma_i^u} x\,dy \le h$. Similarly, $-h \le \oint_{\gamma_i^d} x\,dy \le 0$. Therefore,

$$0 \le \sum_{i=1}^{v} \oint_{\gamma_i^u} y\,dx \le h \cdot v \quad \text{and} \quad -h \cdot v \le \sum_{i=1}^{v} \oint_{\gamma_i^d} y\,dx \le 0.$$

Applying these bounds to the rightmost expression in (2) we see that $-h \cdot v \le A \le h \cdot v$, so that $|A| \le h \cdot v$.

Let R be an $h \times v$ rectangle and let r be the curve traversing its boundary counterclockwise. As r is made up of the same number of length 1 line segments as γ, $\|\gamma\| = \|r\|$. Since R is a polyomino of area $h \cdot v$, [6] applies and $\|r\| \ge 2\lceil 2\sqrt{h \cdot v}\rceil$. Summarizing,

$$\|\gamma\| = \|r\| \ge 2\lceil 2\sqrt{h \cdot v}\rceil \ge 2\lceil 2\sqrt{|A|}\rceil.$$

This completes the proof. □

If $w = \prod_{v=1}^m x_{i_v}^{\epsilon_v}$ is a word in $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$ for which the signed counts of the x_i and the x_j are both zero, then $\|\gamma_{ij}(w)\|$ is the same as the length of the word w after deleting all letters other than $x_i^{\pm 1}$ and $x_j^{\pm 1}$, while $e_{ij}(w) = \oint_{\gamma_{ij}(w)} y\,dx$ by Proposition 7. Thus, Lemma 11 has the following corollary.

Corollary 12. *Let $w = \prod_{n=1}^m x_{i_n}^{\epsilon_n}$ be a word in $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$. Fix some $i \ne j \in \{1, \ldots, n\}$ and assume the signed counts of the x_i and the x_j are both zero. If $e_{ij}(w) = A$, then $|w| \ge 2\lceil 2\sqrt{|A|}\rceil$.*

We are now ready to prove Theorem 5, which gives a lower bound on $C(L)$ in terms of $\mu_{ijk}(L)$.

Theorem 5. *Let $L = L_1 \cup L_2 \cup L_3$ be a 3-component link with vanishing pairwise linking numbers. Then $C(L) \ge 2\lceil 2\sqrt{|\mu_{123}(L)|/3}\rceil$.*

Proof. Let L be a 3-component link with vanishing pairwise linking numbers and F be a C-complex bounded by L. Let $C(F)$ be the number of clasps between the components of F. Let $w_1 = w_1(F)$, $w_2 = w_2(F)$ and $w_3 = w_3(F)$ be the resulting clasp words. Each clasp corresponds to a letter in two of these words, and so

$$2C(F) = |w_1| + |w_2| + |w_3|.$$

Let $e_1 = e_{23}(w_1)$, $e_2 = e_{31}(w_2)$, and $e_3 = e_{12}(w_3)$. Then $\mu_{123}(L) = e_1 + e_2 + e_3$. Assume without loss of generality that $|e_1| \le |e_2| \le |e_3|$. Then it must be that $|e_3| \ge |\mu_{123}(L)|/3$. Corollary 12 concludes that $|w_3| \ge 2\lceil 2\sqrt{|e_3|}\rceil \ge 2\lceil 2\sqrt{|\mu_{123}(L)|/3}\rceil$.

Now, each letter of w_3 corresponds to either a clasp in $F_3 \cap F_1$ or a clasp in $F_3 \cap F_2$. Each of these clasps produces a letter in w_1 or in w_2. As a consequence $|w_3| \le |w_1| + |w_2|$. Putting this together,

$$2C(F) = |w_1| + |w_2| + |w_3| \ge 2|w_3| \ge 4\lceil 2\sqrt{|\mu_{123}(L)|/3}\rceil.$$

The proof is now completed by dividing by 2. □

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